The role of geometric content in Euclid's diagrammatic reasoning

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Mathematical proofs as found in papers, textbooks and classrooms fall well short of the precise formal conception of proof furnished by modern logic. How is this this disparity to be understood? Two starkly contrasting positions on the question have recently been defended. Yehudi Rav and Hannes Leitgib, cite the disparity as evidence against the formal conception as an account of mathematical reasoning. (See [9], [10] and [6].) Jody Azzouni, on the other hand, aims to explain the disparity from a formalist perspective. (See [2] and [4].) On his account, informal proofs count as proofs because they indicate a fully explicit derivation within a formal system. And so, though informal proofs do not manifest all the features demanded by the formal conception, they are ultimately subject to these demands. The rules of the underlying formal framework system determines what is and is not a proof.

My general aim in this paper is to show the possibility of a third, middle position on the nature of informal proof, where the two opposing views are in a sense combined. My case for its possibility consists in an account of the informal proofs of Euclid's elementary geometry, an account based on recent formalizations of Euclid's diagrammatic proofs. The account confirms the Rav/Leitgib view in that it identifies an irreducible role for geometric content in the reasoning. Yet the role played by geometric content is highly constrained, and the constraints are explicated in formal terms. Inferences grounded on an understanding of geometric content occur within a sharply defined formal structure. Accordingly, geometric proof is relativized to a formal framework, a feature that Leitgib identifies with formal proofs. What is and isn't provable is relative to the formal syntax of the framework and the rules for how it can be used.

The formalizations are those described in [8] and [1]. Being formalizations, they would prima facie seem to support Azzouni's formalist postion. After sketching the relevant features of the formalizations in the first section of the paper, I present an alternative interpretation of them in the second section whereby they serve to characterize the informal, contentual inferences of Euclid's proof method. In the third section, I go on to distinguish the resulting picture of Euclid's proofs from Rav's and Leitgib's general picture of informal proof. I show in particular how the method still imposes formal constraints despite allowing certain types of contentual inferences.

1 The Eu and *E* formalizations of Euclid's diagrammatic proofs

The formalizations of both [8] and [1] stem ultimately from Ken Manders' seminal paper [7]. Manders' general philosophical concern in the paper is with mathematical justification—i.e. what is required for it, and the nature of the techniques developed to meet these requirements. His specific topic is the role of diagrams in Euclid's *Elements*. That the latter has anything to do with the former runs counter to the dominant modern view of diagrammatic arguments. But, as Manders points out, the *Elements* was regarded for most of its history as the embodiment of rigorous mathematical reasoning. This fact calls for an explanation, if we desire a complete account of mathematical inference and justification. Manders seeks to provide one by investigating how exactly Euclid employs diagrams in his arguments.

The result is a compelling analysis which reveals that diagrams have a principled, theoretical role in the *Elements*. Only a restricted range of a diagram's properties are permitted to justtify inferences for Euclid, and these self-imposed restrictions can be explained as serving the purpose of mathematical control. The aim of the work in both [8] and [1] is to build on Manders' insights, and precisely characterize the mathematical significance of Euclid's diagrams in a formal system of geometric proof. The formal system of the [8] is termed **Eu**, and that of [1] is called *E*. (In what follows, when I wish to refer to both, I use the term 'E-systems.')

At the heart of Manders' analysis is the distinction between the exact and co-exact properties of a diagram. Roughly, a diagram's exact properties are those that concern, or depend upon, relations between magnitudes in the diagram, and its co-exact properties the spatial relations of containment and position that are stable across 'some range of every continuous variation' of the diagram. For example, the exact properties of the diagram of figure 1 include the equality of segments AB and DC and the equality of angles ABD and DCF. Its co-exact properties include the containment of segment DE in segment AF, and the intersection of segments DC and EB. What one discovers, if one looks the *Elements* closely, is that Euclid only allows the co-exact properties of a diagram to ground the inference in a proof. To

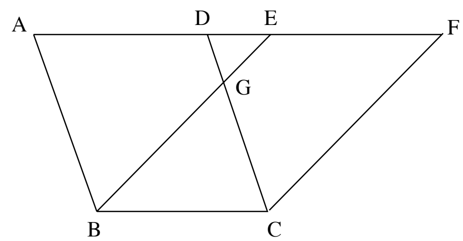

Figure 1:

establish an exact relation, Euclid carries out an argument in the text. That this is the case ought not to be surprising on reflection. Diagrams do not reliably depict exact relations. No matter how precise our techniques for the production of a diagram satisfying exact conditions, there will always be a range of variation in what's produced. Euclid can be understood to account for this by restricting himself to a diagram's co-exact properties.

The E-systems confirm Manders' observations by formalizing Euclid's proofs along two tracks—a diagrammatic one and a sentential one. The former records co-exact information about the configuration under consideration in a proof, while the latter records exact information of the configuration. Within both formal derivations can be produced that match Euclid's proofs closely, often step for step. The main difference between them is the formal nature of the diagrammatic track in each. **Eu** possesses a diagrammatic symbol type which is intended to model the particular physical diagrams that can be used in a proof. In contrast, E models the *information* directly extracted from particular spatial diagrams. This amounts formally to list of primitive relations recording co-exact information. Though potentially important with respect to certain philosophical questions[1], this difference is not relevant to what is of interest here. Both provide the same general picture of Euclid's diagrammatic inferences.

This picture goes beyond Manders' in specifying a method for distinguishing the general from the particular in a representative diagram. Simply recognizing the restricted role of the diagram for Euclid is not enough to remove all concerns about the mathematical rigor of his diagrammatic method. Such concerns arise when we consider diagrams used in proofs in the *Elements* with a construction stage. The construction stage in a Euclidean proof dictates how new geometric elements are to be built on top of a

[1]For a discussion of such philosophical questions, see [5]

given configuration. A demonstration stage then follows, in which inferences from the augmented figure can be made. In the presentation of the proof, the building up process is not shown explicitly. All that appears is the end result of the construction on a particular configuration.

The soundness of Euclid's co-exact inferences from such diagrams is not obvious. The construction is always performed on a particular diagram. Though the diagram is representative of a range of configurations—i.e. all configurations with the same co-exact properties—it cannot avoid having particular exact properties. And these exact properties can influence what co-exact relations come to be exhibited within the final diagram. When the same construction is performed on two diagrams which are equivalent with respect to their co-exact features (hereafter *c.e. equivalent*) but distinct with respect to their exact features, there is no reason to think that the two resulting diagrams will be c.e. equivalent. This leads to doubts that the co-exact relations Euclid expects us to read off of an augmented diagram hold for all possible constructions. And there is nothing in the diagram itself to remove these doubts. The main challenge in formalizing Euclid's diagrammatic method, then, is coming up with rules which license the diagrammatic inferences needed for Euclid's proofs *and* ensure the geometric generality of all such inferences.

Figure 2 provides some examples. The diagrams on the left are initial diagrams, and the diagrams on the right are what results from applying one of more construction steps to the initial diagram. The first, simple example brings out why there is the uncertainty in question. The initial diagram instantiates the assumption, common in the propositions of the *Elements*, that the given configuration under is a triangle. One construction step yields the diagram to the right of it. The step is: construct the perpendicular to the base from the triangles vertex C. Since the angle at the vertex of the triangle is an acute angle, the perpendicular falls outside the triangle. But an obtuse triangle could also have served as an initial diagram, in which case the perpendicular would have fallen inside of it. The second and third examples are diagram pairs representing constructions from proofs in the *Elements*. The second represents the construction for I,2, while the third represents the construction for I,16. There are positional relations displayed in both diagrams which Euclid uses to ground an inference. Yet there is nothing in the diagrams, by themselves, which distinguishes these relations from those that are artifacts of the particular initial diagram chosen to represent the construction.

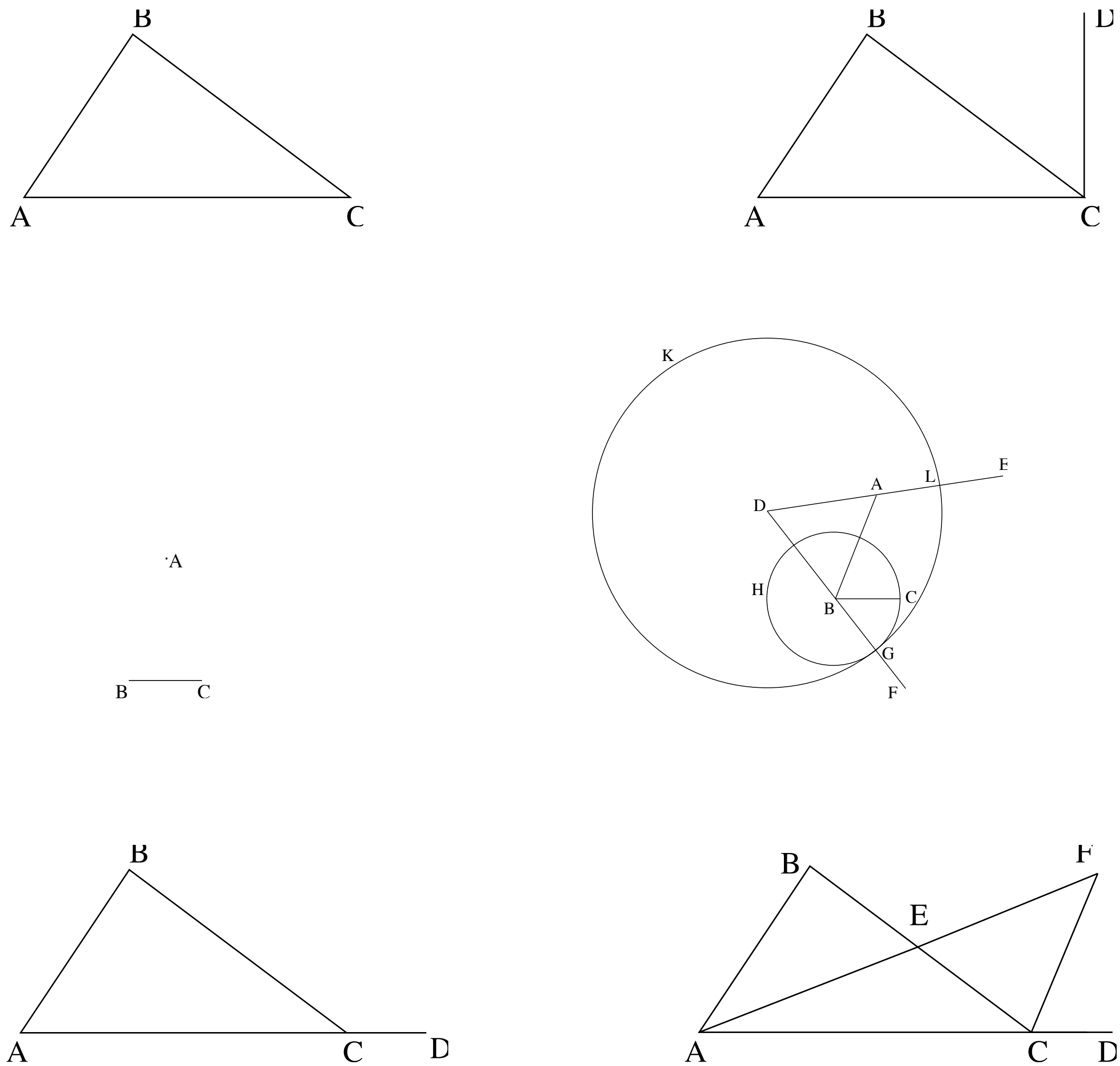

Figure 2: Some initial and augmented diagrams

Under the assumption that the role of diagrams is to record co-exact information, the problem in general terms is the following. In augmenting an initial diagram with new objects according to a geometric construction, a host of new positional relations fall out. Every added point has a position with respect to the pre-existing lines and circles in the diagram. Every added line or circle introduces a new partition of the plane to which all the pre-existing objects have some determinate relation. Nothing in the bare appearance of the diagram separates those relations that hold in general from those that depend on the diagram's particular features. The critical question is: what is the process of reasoning that makes the separation?

The main idea behind the answer provided by the E-systems is that the augmented diagram is to be interpreted in terms of the dependencies induced by the construction. A construction step (e.g. the joining of two points in a segment) produces an object y (e.g. a segment) from a tuple of objects $\vec{x}$ (e.g. two points). If y and the objects of $\vec{x}$ are so related, say that y *directly depends* on the objects of $\vec{x}$. The relation is asymmetrical, and so naturally understood as an ordering relation. We can extend it to all objects of the augmented diagram by taking its transitive closure, and so obtain a partial ordering $\triangleright$ that records the dependencies of the diagrams objects. To describe how this partial ordering enters into the interpretation of the diagram, it is useful to define one more term. Define the relation *is linked* as the symmetrical version of 'directly depends'—i.e. x and y are linked if x directly depends on y or y directly depends on x.

Accordingly, the problem of isolating the general in an augmented diagram comes down to determining which of the unlinked objects in the diagram display general relations. The relations exhibited by linked objects are automatically general. It is built into the concept of a construction step that the constructed object y has a certain positional relation to the objects $\vec{x}$ it was constructed from. If a line segment l is constructed from points p_1 and p_2, for example, then it is immediate that l passes through p_1 and p_2. Also, if a circle c is constructed from center p and radius r, it is immediate that c contains p and r.[2] Generality cannot be said to be built into the relations

[2]These two constructions are postulated by Euclid, but a construction step does not have to be an instance of one of Euclid's postulates for it to induce the relation of direct dependence. A construction which is the application of a proven problem can also induce the relation. So, for instance, if one applies proposition I,1 to a segment l to construct a triangle on that segment, the elements of the triangle distinct from l are directly dependent on l, and thus linked.

exhibited by unlinked objects in a similar way, however. And so, it is indeterminate whether these relations depend on the particular features of the diagram or not. The rules of the E-systems that are of central interest are those that remove this indeterminacy for a certain relation and classify the relation as general.

Because the two systems model diagrams differently, the particular nature of the rules in each is different. In **Eu**, the rules whereby a relation in a particular diagram is distinguished as general are termed *positional* rules. The partial ordering $\triangleright$ is an explicit part of the system. It constrains the application of the positional rules to a particular diagram. The objects at the bottom of the poset are those making up the proofs initial diagram, and so exhibit relations understood as general from the outset. The relations at this level constitute the initial data for the positional rules. One then goes up along the paths of the poset via the positional rules to establish the generality of a relation between unlinked objects. (For an illustration of how this works with **Eu**'s representation of proposition I,2, see [8].) In contrast, in *E*, the ordering $\triangleright$ has no formal role. An *E* diagram in a proof "is nothing more than the collection of generally valid diagrammatic features that are guaranteed by the construction." At the beginning of a proof the collection consists of the relations at the bottom level of $\triangleright$. Each construction step then adds relations with new objects, where the added relations are those that obtain between directly linked objects. This list is then closed under what are termed the *diagrammatic inference rules* of *E*. Any relation in the list generated by this procedure is, according to *E*, something that one is justified as reading off a particular diagram as general.

2 Euclid's contentual diagrammatic inferences

Thus within the E-systems one can follow a formal procedure that grounds Euclid's diagrammatic inferences as general. What philosophical conclusions are to be drawn from this? Specifically, how do the formalizations relate to the informal proofs we see in the *Elements*?

Azzouni's account of mathematical proof would seem to provide a natural answer to the question. On this account, the justificatory force of an informal proof lies in its connection to a formal derivation, occurring in one of many interlocking formal frameworks. The frameworks are individuated by the fundamental formal principles by which derivations are generated. Their particular character defines a mathematical subject. In the paradigm case, they are the axioms and rules of logic which define what can be derived in a first-order theory. They need not, however, fall under this paradigm. There is for Azzouni no constrant on a formal framework's fundamental principles other than that be formal in the precise sense of Turing: there must be an effective way to check whether or not a sequence of the framework's basic formal elements is a derivation. Azzouni does not, and of course cannot, claim that these principles are transparent to the mathematicians who abide by them. He recognizes that the biggest challenge in working out his account is explaining the relation of the underlying formal principles to the informal proofs of practice. (For the most recent way he meets this challenge, see [4]).

Nevertheless, the closer a collection of informal mathematical proofs can be shown to be to the derivations in a formal system, the more plausible Azzouni account becomes, at least for the given collection of mathematical proofs. And so one can easily imagine a proponent of the account to cite the E-systems as evidence for it vis-a-vis Euclid's proofs. Both are formal systems in the precise sense of Turing. And the derivations in both closely match the informal proofs of the *Elements*. These considerations are further reinforced by the fact that in [3] Azzouni himself proposes that Euclid's proofs be thought of as occurring in a formal, diagrammatic system of proof. Much of what he has to say in [3] about diagrams as formal proof symbols echoes Manders' exact/co-exact distinction. Though he touches upon the question of diagrams and generality, he does not fully explore it. The E-systems could thus be thought to fill out this part of Azzouni's discussion, and provide in full detail an Azzounian formal framework for the *Elements*.[3]

Reflection on the rules of the E-systems for ensuring generality speaks against this interpretation, however. The particular character of the rules do not sit comfortably with the role Azzouni assigns to formal frameworks. Specifically, they do not sit comfortably with the idea that formal frameworks define *closed* formal environments within which all mathematical questions are settled. A central goal of Azzouni's project is to show that reference to extra-formal objects and concepts is not necessary to explain mathematical proof. For him everything relevant about a concept used in a proof is built into the proof's underlying formal framework. In this way an Azzou-

[3]It is not clear if Azzouni himself would endorse such an interpretation of **Eu** and *E*. In [3], he describes Euclid's proof method as a 'blend' of a diagram based system and a language based one. The rules of the E-systems that secure generality could on his view be part of the language based component.

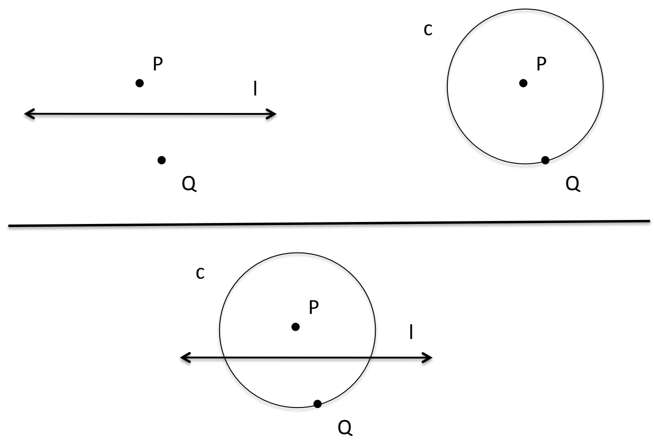

Figure 3: Diagrammatic representation of inference 1

nian framework is closed: nothing from outside the framework can have any bearing on what can or cannot be proven.

To see the implausibility of understanding the E-systems in these terms, consider the following geometric inference

$$\text{Inference 1}\quad \frac{\begin{array}{l}\text{Points } P \text{ and } Q \text{ are opposite sides of lines } l.\\ \text{Circle } P \text{ is the center of circle } c \text{ and } Q \text{ lies on } c.\end{array}}{c \text{ and } l \text{ intersect in two points.}}$$

Figure 3 gives a diagrammatic representation of the inference. The inference occurs in Euclid's proof of proposition 10 in book I. It finds a representation in the E-systems, but not as a basic rule (i.e not as positional or diagrammatic inference rule). A rule used in its derivation corresponds to the following geometric inference:

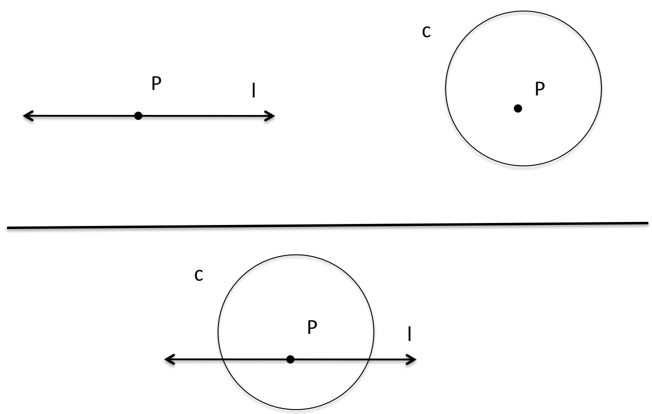

Figure 4: Diagrammatic representation of inference 2

Inference 2

Point P lies in circle c
Point P lies on line l.
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c and l intersect in two points.

(See figure 4 for the inference's diagrammatic representation.) This inference does seem basic, but no more basic than the first one. From a geometric perspective, there seems to be no reason to derive the first from the second. If we accept the second without proof, we should also be willing to accept the first as well. Yet if the E-systems determine what is and isn't a proof, we are obligated to prove the first from the second.

In such cases, the demands of the E-systems, understood as closed formal systems, seem excessive and artificial, an artifact of an arbitrary choice concerning which of Euclid's diagrammatic inferences are to be fixed as basic. Because of this arbitrariness, there can be a mismatch between entitlement to a derivational step and the entitlement to a geometric inference. For the second inference above, for instance, one can doubt that a derivation of an inference from a finite set of rules exists, but there seems to be no room to doubt its geometric soundness. Consequently, the positional rules of **Eu** and the diagrammatic inference rules of E are not convincing as principles which justify proof steps.

I propose instead to interpret them as the formal image of diagrammatic inferences which are grounded in extra-formal geometric concepts. As the two proof systems model diagrams in different ways, the formal image provided by **Eu** highlights different aspects of than that provided by E. With **Eu**, the spatial character of the inferences is brought out. With E, their logical character is. Both however can be understand to realize the following schema (called *ADI* for augmented diagram inference):

$$ADI\ \frac{R_1(\vec{f},\vec{x}) \quad C(\vec{x},y)}{R_2(\vec{f},y)}$$

$R_2(\vec{f},y)$ in the schema denotes the positional relation in the diagram inferred as general. $C(\vec{x},y)$ denotes a basic construction step, which produces y from the tuple $\vec{x}$, and $R_1(\vec{f},\vec{x})$ denotes a collection of positional relations established as general between objects $\vec{f}$ and $\vec{x}$. (In **Eu**, R_1 and R_2 are formally represented as **Eu** diagrams; in E they are represented as a list of diagrammatic primitives.) The tuple $\vec{f}$ is best understood as a frame of reference. $R_1(\vec{f},\vec{x})$ is the condition that the objects of $\vec{x}$ have a certain position with respect to $\vec{f}$. This position along with the nature of the construction $C(\vec{x},y)$ licenses the inference $R_2(\vec{f},y)$—i.e. y has a certain position with respect to the frame of reference $\vec{f}$. What's behind the inference, specifically, is the recognition of a positional invariant. $R_1(\vec{f},\vec{x})$ does not fix the *exact* position of $\vec{x}$ with respect to $\vec{f}$ but specifies a range of positions over which $\vec{x}$ can vary. This limitation in $\vec{x}$'s positional relationship to $\vec{f}$ forces a positional relationship of y to $\vec{f}$.

For an example consider inference 1. In this case, f is the line l, $\vec{x}$ is the point pair (P,Q). The construction C is of circle c with center P and radius $\overline{PQ}$, and the position of P and Q with respect to l forces the intersection of c and l.[4]

[4]It is worth noting that the legitimacy and fundamental nature of inferences of this form is attested by the fact that they appear as axioms in modern axiomatizations of elementary geometry. Pasch's axiom (so named because Pasch first formulated it) and

That a diagrammatic inference has an *ADI* form does not guarantee, of course, that the inference is sound. There are indefinitely many ways to define a frame of reference $\vec{f}$ in terms of line segments and circles as well as indefinitely many possible construction steps $C(\vec{x}, y)$, and so indefinitely many ways to position a construction $C(\vec{x}, y)$ with respect to a frame of reference $\vec{f}$. Some of these force a positional invariant, others do not. On the account of I am advancing, Euclid's method is informal in that it requires of its practitioners the ability to distinguish between the two possibilities. Once an *ADI* has been recognized as sound, it can be fixed as a formal rule and the application of the rule can be formalized, as the E-systems demonstrate. But the recognition of an *ADI* as sound does not, it seems, admit of formalization. This requires understanding points, lines, and circles as spatial forms which can vary continuously. One must understand the range of variation depicted a diagram (e.g. all the possible positions of P and Q on opposite sides of l) and the independence of certain features of the diagram (e.g. the intersection of c and l) from this variation. Practitioners of Euclid's proof method must not only deal with diagrams as formal objects, but also must be able to interpret them geometrically. Accordingly, the proof method emerges as a quasi-formal one. The method's formal framework has a window through which geometrical concepts can enter into proofs.

It is possible that the E-systems could be modified so that they are in a sense diagrammatically complete. This would amount to a formal characterization of *all* sound *ADI*'s and a stipulation of them as basic.[5] It is

axiom $A13'$ in Tarski's theory $\mathcal{E}_2''$ [11] are examples. In words, the Pasch axiom is 'any line constructed from a point on one side of the triangle intersects one of the triangles two other sides.' It is thus a proposition of the form

$$\forall \vec{f}, x, y \ \ (R_1(\vec{f}, x) \ \& \ C(x, y)) \longrightarrow R_2(\vec{f}, y)$$

where $\vec{f}$ is a triangle, R_1 expresses that the point x lies on a side of the triangle, $C(x, y)$ expresses that the line y is constructed from x, and R_2 expresses that the line y intersects one of the triangles other two sides. Tarski's $A13'$ is a proposition of the same form which expresses the geometric soundness of inference 2.

[5] E is shown in [1] to be complete but not diagrammatically complete. That is, it is shown that one can derive in E every truth of Euclidean plane geometry that is representable in E (where 'truth of Euclidean plane geometry' is understood in terms of the models of a first order theory of Euclidean plane geometry). What is not shown is that *all* sound diagrammatic inferences are actually characterized by the formal system as diagrammatic inferences. In other words, it is an open question whether or not there is a geometric inference of the same character of inferences 1 and 2, but requires a derivation

important to emphasize that such an event would not undermine the motivation for understanding Euclid's method as quasi-formal. The point is not that all sound diagrammatic inferences cannot in principle be enumerated formally. The point is that the justification of the inferences is grounded for Euclid in something informal—i.e. an understanding of the geometric concepts diagrams represent. If this understanding could be cleanly codified in a collection of general formal principles, then a case could be made that Euclid's proof method is in fact formal. But no such general formal principles exist. Instead what exists are a multitude of specific ones that are interderivable. My proposal is to understand them as legitimate mathematical judgements, triggered by the construction of a diagram in a proof, and grounded in meaningful geometric concepts.

3 Euclid's formal constraints

Having explicated what I take to be informal in Euclid's proof method, I want in this last section to consider, briefly, how the resulting picture relates to Rav and Leitgib's discussions of informal proof.

On the formal/informal distinction, Rav writes in [9]

> it has been suggested to name Hilbert's Thesis the hypothesis that every conceptual proof can be converted into a formal derivation in a suitable formal system: proofs on one side, derivations on the other, with Hilbert's Thesis as a bridge between the two.

Leitgib presents the formal/informal divide in [6] as a table of contrasting features:

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using rules other than the diagrammatic inference rules.

Formal Provability	Informal Provability
formal syntax	not syntactically determined
logical level: 1st order, 2nd order,. . .	not logically determined
terms, formulas	interpreted terms, interpreted formulas
logical rules, syntactically encoded	truth preservation, evident steps
logical axioms	no logical axioms
mathematical axioms	lack of axioms
	axioms as partial denitions
	deductive "gaps" vs.
	true and evident foundational axioms

Both thus frame the formality or informality of a proof is all or nothing affair. One is either on one side of Hilbert's bridge or the other. Either every step in a proof is tied to explicit formal rules, or the full mathematical meaning of a proof's terms is available at any stage of a a proof. My aim in this last section is to complicate this picture. Specifically, I show that the informality of the inferences described in the last section does not render the formal framework in which they occur idle. Though the framework allows geometric content into a proof, it restricts its use. Inferences that arguably follow from the meaning of geometric concepts are disallowed. Accordingly, the method is in fact justifiably termed 'quasi-formal,' rather than simply 'informal.' In terms of Leitgib's table, the method has a formal syntax and syntactically encoded rules that non-trivially constrain what does and does not count as a proof.

This can be seen by looking at proposition 20 of book I. The proposition asserts the triangle inequality: the sum of any two sides of a triangle is greater than a third. Commenting on this proposition, Proclus refers to the Epicureans who ridiculed Euclid for taking the trouble to prove something that was known even to an ass. And indeed, the proposition seems an immediate consequence of the concept of a straight line. Yet if the account I am proposing of Euclid's method is correct, it cannot for Euclid be asserted without proof. Inequalities among magnitudes in a configuration can only be asserted without proof if one magnitude contains another in a diagram. For magnitudes where no such containments obtain—as with the sides of a triangle—one must show via a diagram that either:

- the lesser magnitude X is equal to a magnitude X' that is contained in the greater one Y

- the greater magnitude Y is equal to a magnitude Y' that contains the lesser one X.

As diagrams on the account are formal, syntactic objects, subject to formal rules, this criteria of proof is in part a formal one.

Specifically, the E-system rules fix a syntactic criteria for what 'showing via a diagram' is. One cannot simply take a diagram with spatially separated lengths AB and CD, place a point B' on CD and assert that $CB' = AB$ in order to prove that $AB < CD$. This would amount to *explaining* what is to be proved by the method; it would not be a proof. A proof requires a construction of the point B' according to the rules for constructing diagrams, and a sound diagrammatic inference that B' lies on CD. The latter requirement is where geometric concepts have a role, and hence where there is an openness to the method from a formal perspective. The former requirement, however, is a formal one. The syntactic rules for constructing diagrams limit the way geometric content can be called upon.

The following argument for the triangle inequality, which is not a permissible one by the method, illustrates this.

Given triangle ABD (figure 5), suppose $A'B'$ and $B'D$ initially coincide with AB and BD, and continuously move A' in the direction of the ray $\vec{CA}$ so that B' lies on the ray. $A'B' + B'D > AD$, $AB = A'B'$ and $A'D = AD$, so $AB + BD > AD$.

QED

The argument augments the initial triangle ABD with points A' and B'. The inference that $A'B' + B'D > AD$ is based on the position of B' on the line $A'D$ and the containment of AD by AD'. That these positional relations hold for any position of B above the segment AD, is a diagrammatic inference. Assuming the legitimacy of the construction of A' and B', the inference seems no less obvious from a geometric perspective than inferences 1 and 2 above. It seems to draw from the same general geometric notions of continuity and invariance. The construction of A' and B' is not however legitimate. The rules for augmenting a diagram do not allow one to produce points by moving segments continuously as in a linkage. In particular, the method only allows one to reason about continuity and invariance with respect to diagrams produced via line and circle constructions.

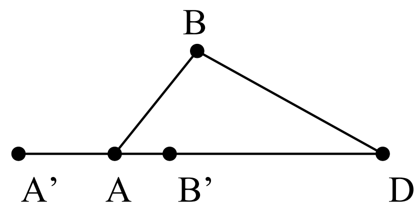

Figure 5: Augmented diagram for proposed proof of I, 20

The upshot is that what is provable by the method and what is obvious from the concepts are not the same, and the division between them can be accounted for by the rules of the E-systems. If this account is correct, a variety of questions arise. What is served by following a quasi-formal method, rather than an informal one? Do the Euclid's self-imposed restriction serve some principled epistemological goal, or are they simply conventional? Broadening our view from ancient geometry to mathematics as a whole, we can ask whether the quasi-formal character of Euclid's method is a feature of other mathematical contexts. These are questions, however, for future work. My aim here was simply to flesh out the possibility of a method of proof that does not sit on either side of Hilbert's bridge, but on it.

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