Ensuring Generality in Euclid's Diagrammatic Arguments

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Abstract. This paper presents and compares **FG** and **Eu**, two recent formalizations of Euclid's diagrammatic arguments in the *Elements*. The analysis of **FG**, developed by the mathematician Nathaniel Miller, and that of **Eu**, developed by the author, both exploit the fact that Euclid's diagrammatic inferences depend only on the topology of the diagram. In both systems, the symbols playing the role of Euclid's diagrams are discrete objects individuated in proofs by their topology. The key difference between **FG** and **Eu** lies in the way that a derivation is ensured to have the generality of Euclid's results. Carrying out one of Euclid's constructions on an individual diagram can produce topological relations which are not shared by all diagrams so constructed. **FG** meets this difficulty by an enumeration of cases with every construction step. **Eu**, on the other hand, specifies a procedure for interpreting a constructed diagram in terms of the way it was constructed. After describing both approaches, the paper discusses the theoretical significance of their differences. There is in **Eu** a context dependence to diagram use, which enables one to bypass the (sometimes very long) case analyses required by **FG**.

Each of the arguments in Euclid's *Elements*, as given in [1], comes equipped with a geometric diagram. The role of the diagram in the text is not merely to illustrate the geometric configuration being discussed. It also furnishes a basis for inference. For some of Euclid's steps, the logical form of the preceeding sentences is not enough to ground the step. One must consult the diagram to understand what justifies it. Consequently, Euclid is standardly taken to have failed in his efforts to provide exact, fully explicit mathematical proofs. Inspection of geometric diagrams is thought to be too vague and open-ended a process to play any part in rigorous mathematical reasoning.

This assumption has recently been disproved. **FG**, developed by Nathaniel Miller and presented in [3], and **Eu**, developed by the author in [4], are formal systems of proof which possess a symbol type for geometric diagrams. Working within each system, one can reconstruct Euclid's proofs in an exact and fully explicit manner, *with* diagrams. In this paper I compare the two systems as accounts of Euclid's diagrammatic reasoning. In the first part, I discuss the feature of Euclid's diagram use which makes both formalizations possible. In the second, I explain how both systems codify this use in their rules. In **FG**,

G. Stapleton, J. Howse, and J. Lee (Eds.): Diagrams 2008, LNAI 5223, pp. 222–235, 2008.

the content of single diagram is context independent, and the result of a geometric construction is a disjunctive array of such diagrams. Alternatively, in **Eu**, the content of a diagram depends systematically on the way it is constructed in a proof. The need for the disjunctive arrays of **FG**, which can become very large, is thus avoided.

1 What a Diagram Can do for Euclid

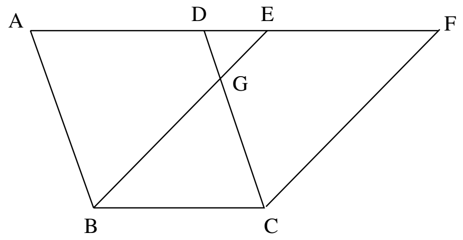
A close reading of the *Elements* reveals that the significance of a diagram in a proof is neither vague nor open-ended for Euclid. The first to discover this is Ken Manders, who laid out his insights on ancient geometric proof in [2].[1]

To explain the division of labor between text and diagram in ancient geometry, Manders distinguishes between the *exact* and *co-exact* properties of geometric diagrams. Any one of Euclid's diagrams contains a collection of spatially related magnitudes—e.g. lengths, angles, areas. For any two magnitudes of the same type, one will be greater than another, or they will be equal. These relations comprise the *exact* properties of the diagram. How these magnitudes relate topologically to one another—i.e. the regions they define, the containment relations between these regions—comprise the diagram's *co-exact* properties. Diagrams of a single triangle, for instance, vary with respect to their exact properties. That is, the lengths of the sides, the size of the angles, the area enclosed, vary. Yet with respect to their co-exact properties the diagrams are all the same. Each consists of three bounded linear regions, which together define an area.

The key observation is that Euclid's diagrams contribute to proofs *only* through their co-exact properties. Euclid never infers an exact property from a diagram unless it follows directly from a co-exact property. Exact relations between magnitudes which are not exhibited as a containment are either assumed from the outset or are proved via a chain of inferences in the text. It is not difficult to hypothesize why Euclid would have restricted himself in such a way. Any proof, diagrammatic or otherwise, ought to be reproducible. Generating the symbols which comprise it ought to be straightforward and unproblematic. Yet there seems to be room for doubt whether one has succeeded in constructing a diagram according to its exact specifications perfectly. The compass may have slipped slightly, or the ruler may have taken a tiny nudge. In constraining himself to the co-exact properties of diagrams, Euclid is constraining himself to those properties stable under such perturbations.

For an illustration of the interplay between text and diagram, consider proposition 35 of book I. It asserts that any two parallelograms which are bounded by the same parallel lines and share the same base have the same area. Euclid's proof proceeds as follows.

[1] The paper was written in 1995 but published only recently, in *Philosophy of Mathematical Practice*, edited by Paolo Mancosu (Clarendon Press, 2008). Despite the fact the paper existed only as draft for most of its 13 years, it has been influential to those interested in diagrams, geometry and proof. Mancosu describes it an 'underground classic.'

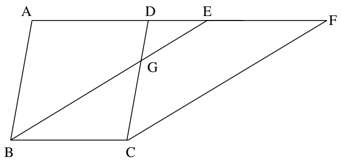


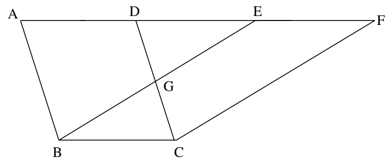
Let ABCD, EBCF be parallelograms on the same base BC and in the same parallels AF, BC.
Since ABCD is parallelogram, AD equals BC (proposition 34). Similarly, EF equals BC.
Thus, AD equals EF (common notion 1).
Equals added to equals are equal, so AE equals DF (Common notion 2).
Again, since ABCD is a parallelogram, AB equals DC (proposition 34) and angle EAB equals angle FDC (proposition 29).
By side angle side congruence, triangle EAB equals triangle FDC (proposition 4).
Subtracting triangle EDG from both, we have that the trapezium ABGD equals the trapezium EGCF (common notion 3).
Adding triangle GBC to both, we have that ABCD equals EBCF (common notion 2).
QED

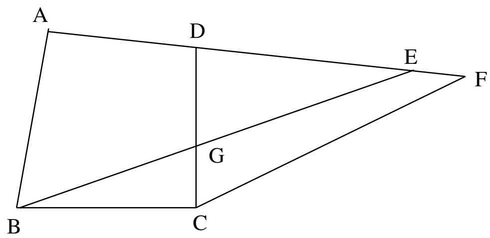
The proof is independent of the diagram up until the inference that AE equals DF. This step depends on common notion 2, which states that if equals are added to equals, the wholes are equal. The rule is correctly invoked because four conditions are satisfied: $AD = EF$, $DE = DE$, DE is contained in AE, and DE is contained in DF. The first pair of conditions are exact, the second pair co-exact. Accordingly, the first pair of conditions are seen to be satisfied via the text, and the second pair via the diagram. Similar observations apply to the last two inferences. The applicability of the relevant common notion is secured by both the text and the diagram. With just the textual component of the proof to go on, we would have no reason to believe that the necessary containment relations hold. Indeed, we would be completely in the dark as to the nature of containment relations in general.

The standard line is that this situation needs to be rectified with something like a betweenness relation. Manders's opposing thesis is that diagrams function in the *Elements* as reliable symbols because Euclid only invokes their co-exact features. Though we may not be able to trust ourselves to produce and read off the exact properties of diagrams, we can trust ourselves to produce and read off co-exact properties. Thus, Euclid seems to be within his rights to use diagrams to record co-exact information. If Manders's analysis is correct, Euclid's proofs ought to go through with diagrams which are equivalent in a co-exact sense (hereafter *c.e. equivalent*), but differ with respect to their exact properties. This turns out to be the case. The proof of proposition 35, for instance, still works if we substitute either of the following for the given diagram.

The diagram need not even satisfy the stipulated exact conditions. The diagram also fulfills the role the proof demands of it. The diagram's burden is to







reveal how certain co-exact relationships lead to others. It is not used to show exact relationships. This is the job of the text. The proof must invariably employ a particular diagram, with particular exact relationships. But since the proof only calls on the co-exact relationships of the diagram, it holds of *all* diagrams which are c.e. equivalent to it.

Manders' observations naturally suggest the general approach of both **FG** and **Eu** in formalizing Euclid. As topological objects, Euclid's diagrams are discrete. What identifies them, topologically, is the way their lines and circles partition a bounded region of the plane into a finite set of regions. Thus, the discrete syntactic objects which are to function as diagrams in a formalization ought to be individuated in the same manner. The first challenge, then, is to define such syntactic objects precisely. The second is to formulate suitable rules for how objects so defined are to be used in proofs.

Though **FG** and **Eu** meet the first challenge in different ways, my focus is on their different approaches to the second challenge. With respect to the goal of defining syntactic objects which express the information Euclid relies on his diagrams to express, the diagrams of **FG** and **Eu** are equally sufficient. **Eu** diagrams are arguably closer to the diagrams of the Euclidean tradition, in that **Eu** lines and **Eu** circles are not purely topological.[2] I will not explore, however, whether this difference between **FG** and **Eu** is a significant one. Similarly, I will pass over the fact that **FG** is a purely diagrammatic proof system, and **Eu**

[2] The only initial constraint on the line and circle segments of **FG** diagrams is that they be one-dimensional. Though the definition of a well-formed **FG** diagram imposes restrictions on how such syntactic elements can relate to one another topologically, it still leaves room for diagrammatic lines and circles which twist and turn in wildly non-linear and non-circular ways. In contrast, **Eu** diagrams possess an underlying array structure, which allows for linearity and convexity to built into the definitions of lines and circles from the start. For the precise formal characterization of diagrams in **FG** and **Eu**, see pp. 21-34 in [3] and pp. 14-40 in [4]. For a discussion which raises doubts about the faithfulness of **FG** 's purely topological diagrams to the Euclidean tradition, see [5].

is a heterogeneous one. Both differences are not essential in understanding the central difference between the proof rules of **Eu** and **FG**.

2 Euclid's Constructions in FG and Eu

The diagram of a Euclidean proof rarely displays just the geometric elements stipulated at the beginning of the proof. They often have a construction stage dictating how new geometric elements are to be built on top of the given configuration. The demonstration stage then follows, in which inferences from the augmented figure can be made. The building up process is not shown explicitly. All that appears is the end result of the construction on a particular configuration.

As the proof of proposition 35 has no construction stage, it fails to illustrate this common feature of Euclid's proofs. The diagram of the proof contains just those elements which instantiate the propositions general co-exact conditions. We are thus justified in grounding the result on the co-exact features of the diagram, given that we only apply the result to configurations which are c.e. equivalent to the diagram.

The soundness of Euclid's co-exact inferences is much less obvious when the proof's diagram contains augmented elements. The construction is always performed on a particular diagram. Though the diagram is representative of a range of configurations—i.e. all configurations c.e. equivalent to it—it cannot avoid having particular exact properties. And these exact properties can influence how the co-exact relations within the final diagram work out. When the same construction is performed on two diagrams which are c.e. equivalent but distinct with respect to their exact features, there is no reason to think that the two resulting diagrams will be c.e. equivalent.

Consider for example the construction of proposition 2 of book I of the *Elements*. The proposition states a construction problem: given a point A and a segment BC, construct from A a segment equal to BC. Euclid advances the following construction as a solution to the problem:

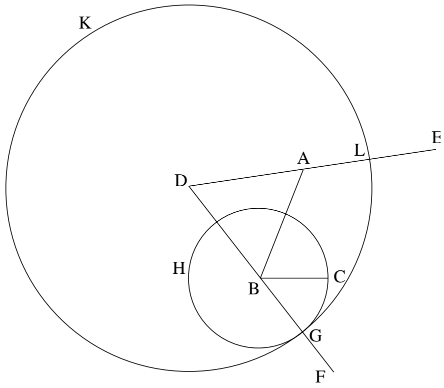
From the point A to the point B let the straight line AB be joined; and on it let the equilateral triangle DAB be constructed.

Let the straight lines AE, BF be produced in a straight line with DA and DB.

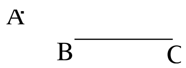
With center B and radius BC let the circle GCH be described; and again, with center D and radius DG let the circle GKL be described.

If the construction is performed on the *particular* point A and a *particular* segment BC

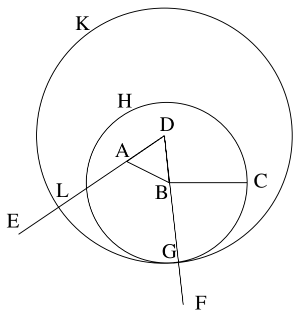
the result is the diagram



If however the construction is performed on the different particular configuration



the result is

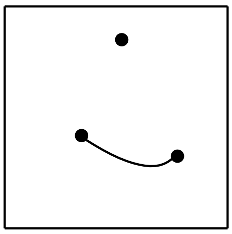


This diagram is distinct from the first diagram, topologically. Euclid nevertheless uses the co-exact features of one such diagram to argue that the construction does indeed solve the stated construction problem. The crucial step in the argument is the inference $AL = BG$. This follows from an application of the equals subtracted from equals rule. And for this to be applicable, A must lie on the segment DL and B must lie on the segment DG. So with these two diagrams we see that with two of the possible exact positions A can have to BC the topology needed for the proof obtains. But *prima facie* we have no mathematical reason to believe that it obtains for *all* the other positions A can have to BC.

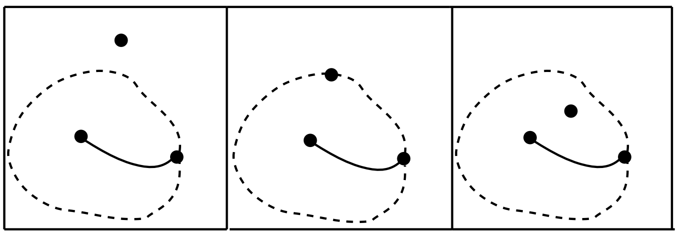
And so, the vexing question is: how do we know that the co-exact features that Euclid isolates as general in a constructed diagram are shared by *all* diagrams which could result from the construction? Though Euclid never mistakes a property particular to an individual diagram as general, he does not provide any explicit criteria for how the separation of the general from the particular is to be made. What the formalizations of **FG** and **Eu** must do, if they are to count as formalizations, is furnish such criteria via its rules of proof.

2.1 The Diagram Arrays of FG

The rules of **FG** do this via disjunctive diagram arrays. A Euclidean construction in **FG** is not carried out via a single, representative diagram, but via an array of representative diagrams. In applying a construction step (such as joining two points in a segment, or drawing a circle on a radius) to a diagram D, one must produce the array representing all topological cases which could possibly result from applying the step to a figure represented by D. For instance, if D is the **FG** diagram



then constructing a circle on the segment of D produces in **FG** the array



As only topological features can be read off from a diagram, D contains no information about the distance of the left-endpoint of the segment to the point off the segment. And so, it is consistent with D that the latter point sit outside, on, or inside the constructed circle.

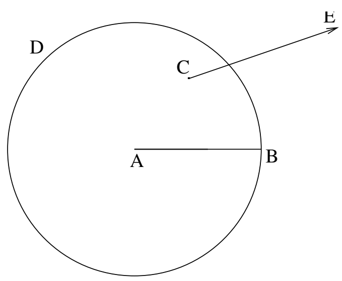
Applying a construction step to an array A produces the array of all diagrams obtained by applying the step to each diagram of A. Thus, the array produced after n construction steps contains all topological cases which could possibly result from those n steps. Logically, the array is similar to a propositional statement in disjunctive normal form. It asserts that the geometric figure of the proof has the properties of one of the diagrams in the array. Once all contradictory diagrams (diagrams whose markings equate the part of a whole to the whole) are thrown out, one is then in a position to discern what holds in general. This consists in those properties manifest in all diagrams of the array.

The central technical achievement of **FG** is the specification of a mechanical procedure which given any initial diagram and any geometric construction as input outputs the appropriate array. Though it is clear what all the cases are when a circle is added to a diagram consisting only of a segment and a point, it is not clear if there is a general method for enumerating cases given any construction step and any diagram. The diagram does not have to become that much more complex for the range of possible cases consistent with a construction step to become obscure. Even if some cases can be seen, one usually lacks a guarantee that these constitute *all* cases to be considered.

The purely topological character of his diagrams, however, allows Miller to specify a method which has such a guarantee built in. The reason that the range of cases which come with a construction step is obscure is that it is not immediate how the metric symmetries of lines and circles restrict what is and isn't possible topologically. Yet once we allow line and circles to bend any which way, the obscurity vanishes. The range of cases emerges as the range of all topological possibilities consistent with the conditions Miller stipulates of the diagrams of his system. These conditions comprise the definition of a *nicely well-formed diagram* in **FG**. They ensure that lines and dotted lines form configurations which behave in a rough topological way like Euclidean lines and circles. One condition, for instance, ensures that two circles intersect no more than twice. And so, specifically, the array in an **FG** formalization of an Euclidean proof consists of all nicely well-formed diagrams obtainable from the proofs construction. These can be generated by a straightforward, if tedious, procedure, implemented by Miller in a computer program named **CDEG** (for Computerized Diagrammatic Euclidean Geometry).

2.2 Diagrammatic Inferences in Eu

The diagrammatic proof method of **Eu** is based on the principle that what is general in a diagram depends on how it was constructed.[3] Consider the diagram



Many distinct constructions could have produced it. For instance, the initial configuration could have been the segment AB, and the construction steps leading to the diagram could have been:

[3] The principle is perhaps close to what Kant is talking about when he speaks of the "the universal conditions of the construction" in the passage quoted above. For a discussion which relates Kant's philosophy of mathematics to Euclid's geometric constructions, see [6].

- draw the circle D with center A and radius AB.
- pick a point C in the circle D, and a point E outside it.
- produce the ray CE from the point C.

Call this construction **C1**. Alternatively, it is possible that the initial configuration consists of the segment AB and the points C and E, while the construction consists of the following two steps:

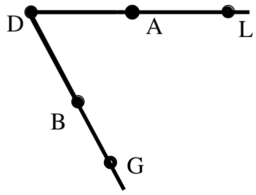
- draw the circle D with center A and radius AB.
- produce the ray CE from the point C

Call this construction **C2**.

Now, if **C1** is responsible for the diagram, we are justified in taking the position of C within D as a general property of the diagram. The act of picking C in D fixes the point's position with respect to the circle as general. And since we know the position of C relative to D is general, we can pick out the point of intersection of the ray CE with D with confidence. It always exists in general, since a ray originating inside a circle must intersect the circle. In contrast, none of these inferences are justified if **C2** is responsible for the diagram. Nothing is assumed from the outset about the distance of the point C to A. And so, even though C lies within D in this particular diagram, it could possibly lie on D or outside it. Further, as the position of C relative to D is indeterminate, the intersection point of CE and D cannot be assumed to exist in general, even though one exists in this particular diagram.

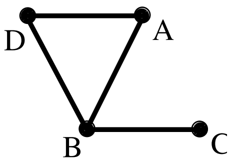
Viewing proposition 2 in this way, we can satisfy ourselves that Euclid's diagrammatic inferences are sound. Though the position of segment BC with respect to the triangle ADB is indeterminate, what that segment contributes to the proof is the circle H, whose role in turn is to produce an intersection point G with the ray DF. The intersection point always exists no matter the position of BC to the ray DF. We can rotate BC through the possible alternatives, and we will always have a circle H whose center is B. And this is all we need to be assured that the intersection point G exists. The ray DF contains B, since it is the extension of the segment DB, and a ray which contains a point inside a circle *always* intersects the circle.

A similar argument shows that the intersection point L of the ray DE and the circle K always exists. The argument does not establish, however, that A lies *between* D and L. Here a case analysis is forced upon us. We must consider the case where A coincides with L, or the case where L lies between A and D. These latter two possibilities, however, are quickly ruled out, since they imply that $DL = DA$ or that $DL < DA$. This contradicts $DA < DL$, which follows from the equalities $DA = DB$, $DG = DL$ and the inequality $DB < DG$. (The equalities follow from the properties of equilateral triangles and circles. The inequality $DB < DG$ is entailed by the fact that B lies between D and G, which holds because G was stipulated to lie on the extension of DB.) Thus, Euclid's construction in I,2 can always be trusted to produce a configuration



where $DA = DB$ and $DG = DL$. Accordingly, the equals-subtracted-from-equals rule is applicable, and we can infer that $AL = BG$.

Thus runs the proof of proposition 2 in **Eu**. Though the informal version given here is much more compact, each of its moves is matched in the formal version. Generally, proofs of propositions in **Eu** are two tiered, just as they are in the *Elements*. They open with a construction stage, and end with a demonstration stage. The rules which govern the construction stage are relatively lax. One is free to enrich the initial diagram Δ_1 by adding points, joining segments, extending segments and rays, and constructing a circle on a segment. Presented as a sequence of **Eu** diagrams, the construction stage for proposition 2 is as follows.

1


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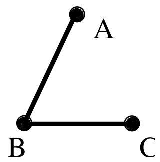

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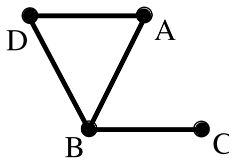

$DB = DA$

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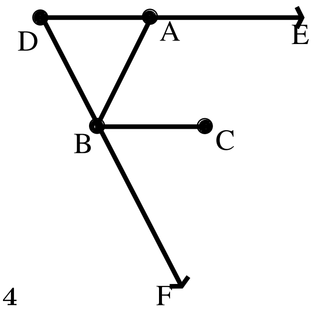

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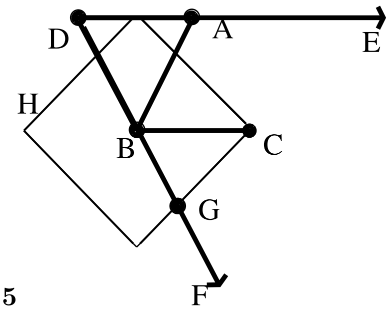

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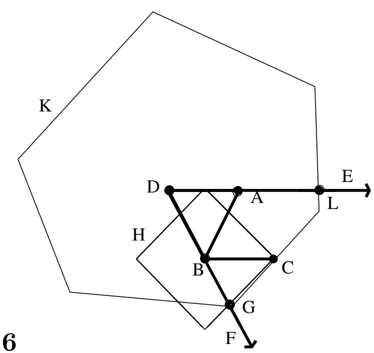
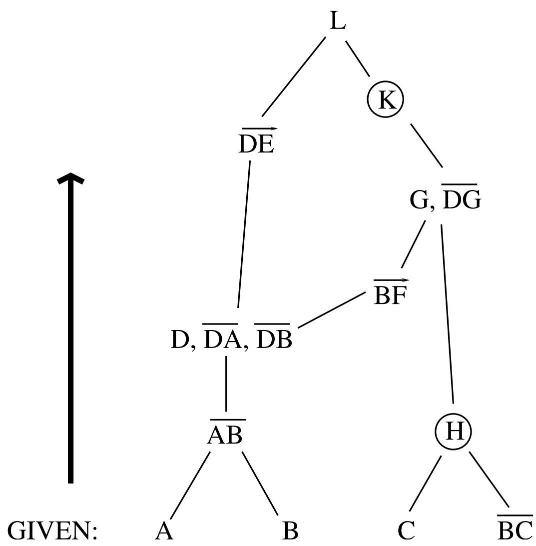


The last step in the construction yields a diagram Σ, which contains all the objects to be reasoned about in the demonstration. But it is not Σ alone, but the whole construction history of Σ, which determines what can be inferred in the demonstration.

The sequence of steps by which Σ was constructed determine a partial ordering $\triangleright$ of its geometric elements. An element x in the diagram *immediately precedes* y if the construction of y utilized x. For instance, if the points A and B are joined in a construction, the points A and B immediately precede the segment AB. Likewise, if a circle H is constructed with radius BC, the segment BC immediately precedes H. The complete partial ordering $\triangleright$ is simply the transitive closure of the *immediately precedes* relation. It serves to record the dependencies among the elements of Σ. For example, for the representation of I,2 in **Eu**, the partial ordering works out to be:



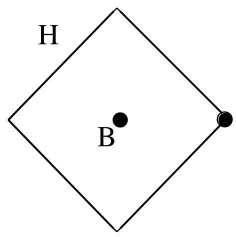
As the elements A, B, C and BC are part of the initial diagram Δ_1, they were not constructed from any elements in Σ, and so nothing precedes them. The rest appear somewhere above these, according to the way they were introduced.

The construction thus produces a tuple

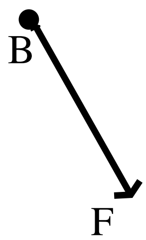
$$\langle \Sigma, M, \triangleright \rangle$$

which is called the *context* of the proof. The term M is the metric assertion which records the exact relationships stipulated from the beginning or introduced during the course of the construction. These three pieces of data serve as input for the demonstration stage. Rules of this stage are of two types: positional and metric.

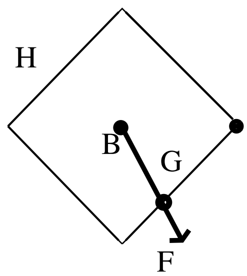
An application of a positional rule results in a sub-diagram of Σ. Deriving a sub-diagram amounts to confirming the generality of the co-exact relationships exhibited in it. As such, the application of the rules are constrained by $\triangleright$. One can introduce as a premise any sub-diagram of the initial diagram Δ_1—i.e. any sub-diagram consisting of elements which have nothing preceeding them by $\triangleright$. Any other sub-diagram must be derived from these by the positional rules, where the derivations proceed along the branches laid out by $\triangleright$. For instance, in the proof of proposition 2, one derives from the segment BC the sub-diagram



and from the points A and B the sub-diagram



From these two sub-diagrams we can then derive



from a rule which encodes the general condition for the intersection of a ray and a circle.

The complete **Eu** formalization of proposition 2 can be viewed on pp. 95-109 of [4]. Both the construction and the demonstration in the formalization contain spatially separated diagrams. They are spatially separated, however, only to make clear the rule being applied at each step. The formalization is intended to model a process where one *single* diagram Σ is being constructed and inspected.

Any given diagram in the **Eu** construction represents a certain point in the construction of Σ. Any given diagram in the demonstration represents the part of Σ being considered at a certain point in the proof. As far as it possible, then, **Eu** formalizes Euclid's diagrammatic reasoning as *agglomerative* (in the sense of the term used by K. Stenning in [7]). This is in contrast to **FG**, which requires a disjunctive array of *different* diagrams every time the topology of a construction is not unique.

Understanding the logical properties of **Eu**, and implementing the system in a computer program, are goals of current and future work. It is straightforward to check the relative consistency of **Eu** to a modern axiomatization of elementary geometry. Claims formulated in **Eu** have a natural interpretation in terms of first order formulas composed of the primitives of such an axiomatization. The soundness of **Eu**'s proof rules is then easily checked in terms of this interpretation. The completeness of **Eu** with respect to a modern formalization is a more difficult question. To address it, a non-diagrammatic system of proof designed to model **Eu**, termed ***E***, has been developed (and will soon be presented in a forthcoming paper by Jeremy Avigad, Ed Dean and myself). The new system can be shown to be complete. That is, it can be shown that a modern axiomatization of elementary geometry is a conservative extension of ***E***. The question of whether **Eu** is complete then reduces to the question of how well it is modelled by ***E***. Work is now being done to implement ***E***. A goal of future research is to implement **Eu** directly—i.e. to develop a program where diagrams are codified just as they are formally defined in **Eu**.

3 Concluding Remarks

From a logical perspective, requiring disjunctive arrays of diagrams as **FG** does is a natural way to ensure generality in a diagrammatic formalization of Euclid. Yet if the goal is to understand what underlies the single diagram proofs of the *Elements*, the approach is far less natural. When a case-heavy **FG** formalization is laid beside Euclid's original version, the original does not appear deficient. Rather, the multitude of cases generated by the rules of **FG** appear excessive. The geometric differences recorded by a case-branching often do not seem material to the issue the proof decides.

This comes out if we compare **FG**'s formalization of proposition 2 with **Eu**'s. In [3] Miller does not explicitly discuss how **FG** handles the proof, but one can work out by hand the cases generated in the **FG** proof, at least for the first few steps of the construction. My efforts yielded 57 cases at the fourth step. Pushing further with the whole construction on three of these yielded 50 more. (At this point my will to continue gave out.) What distinguishes the cases are positional relations which are irrelevant to the inferences Euclid makes later, in that the relations need not be attended to for the soundness of the inferences to be confirmed. As the **Eu** formalization shows, we can focus in on certain relations in a single representative diagram (Σ in the **Eu** formalization) and ignore others in verifying the generality of the result. We do not have to check that the result holds in a long list of cases. That Euclid's proof allows us to do this does not seem to be an accidental feature. It seems, rather, to be a key mathematical insight of the proof. With proposition 2 and others throughout the *Elements*, Euclid seems deliberately to frame his arguments so that it suffices, or almost suffices, to consider a single diagram. The formal account of **Eu** respects this feature of the *Elements* while the formal account of **FG** does not. In **FG**, the one and only way to secure a general result with diagrams is by a brute enumeration of cases.

Eu avoids the need for such enumerations (most of the time) by allowing the content of a diagram to be context dependent. If the same geometric diagram is to communicate the same topological information every time it appears in a proof, then an enumeration of cases seems the only option in formalizing Euclid's diagrammatic constructions. With any faithful formal characterization of Euclid's diagrams there will be instances where what is representative in a diagram depends on how it is constructed. Specifically, there will be diagrams which when understood as a result of one of Euclid's constructions manifest non-representative features, and when understood as the result of another construction manifest nothing non-representative. Since such diagrams can in *some* instances be understood as representing all the topological relations it embodies, it seems ad hoc and overly restrictive to stipulate a partial content of the diagram in *all* instances. And so, if we must stipulate a content for all instances, the only systematic approach which seems available is that of **FG**: have a diagram communicate all topological relations manifest in it, and require case branching with every construction step. What **Eu** shows is that we need not do this. It is possible for the content of a diagram to vary across proofs. The information a diagram holds in a proof can be understood to depend systematically on particular features of the proof. Whether or not it is illuminating to understand diagram use in other mathematical contexts in this way seems a question worth pursuing.

References

1. Heath, T.: The Thirteen Books of Euclid's Elements—translated from the text of Heiberg. Dover Publications, New York (1956)
2. Manders, K.: The Euclidean Diagram. In: Mancosu, P. (ed.) Philosophy of Mathematical Practice, pp. 112–183. Clarendon Press, Oxford (2008)
3. Miller, N.: Euclid and His Twentieth Century Rivals: Diagrams in the Logic of Euclidean Geometry. Center for the Study of Language and Information, Stanford (2007)
4. Mumma, J.: Intuition Formalized: Ancient and Modern Methods of Proof in Elementary Euclidean Geometry. Ph.D Dissertation, Carnegie Mellon University (2006), `www.andrew.cmu.edu/jmumma`
5. Mumma, J.: Review of Euclid and His Twentieth Century Rivals: Diagrams in the Logic of Euclidean Geometry. Philosophica Mathematica 16(2), 256–264 (2008)
6. Shabel, L.: Kant's Philosophy of Mathematics. In: Guyer, P. (ed.) The Cambridge Companion of Kant, 2nd edn. Cambridge University Press, Cambridge (2006)
7. Stenning, K.: Distinctions with Differences: Comparing Criteria for Distinguishing Diagrammatic from Sentential Systems. In: Anderson, M., Cheng, P., Haarslev, V. (eds.) Diagrams 2000. LNCS (LNAI), vol. 1889, pp. 132–148. Springer, Heidelberg (2000)