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Research Statement

The questions motivating my work are the nature of mathematical justification, generally, and the nature of geometric thought and reasoning, specifically. I addressed them in my doctoral thesis by developing an account of Euclid's diagrammatic proofs in the *Elements*. The key idea of this work was the characterization of geometric diagrams as formal symbols of proof. The presence of a diagram in a geometric proof is typically understood to be a mark of informality. My doctoral thesis revealed the error in this received view, and thus challenged the presuppositions behind it—i.e. presuppositions regarding the rigor of diagrammatic reasoning in geometry on the one hand, and the nature and role of formal mathematical symbols on the other.

Since completing the thesis I have worked on sharpening the philosophical challenges my doctoral work poses to received views on mathematical proof and geometry. I have as a result developed a research program exploring a range of interconnected foundational, ontological and cognitive issues. Before describing this program, I first present in greater detail the philosophical perspective my doctoral work motivates. I then lay out the questions I plan to explore.

The **Eu** picture of elementary geometry

The methodology of Euclid's *Elements* is clearly different from that of modern theories of geometry. The precise nature of the differences, and the significance of them, is not however obvious. The aim of my doctoral thesis was to make the differences precise with a formal account, termed **Eu**, which characterizes Euclid's proofs as essentially diagrammatic. **Eu** is a formal proof system that has a diagrammatic symbol type, and rules for governing the use of such symbols in proofs. Within the system one can produce derivations that match Euclid's arguments closely, often step for step. **Eu** thus counters the standard modern assessment of Euclid's methodology as intuitive and un-rigorous, and opens up new possibilities for understanding Euclid, and informal mathematical practice generally, as intuitive and mathematically legitimate.

A major reason for the modern dismissive attitude towards Euclid is the great success of the arithmetization program within mathematics over the past two centuries. The central premise of the program is that discrete, arithmetical notions are fundamental. Continuous, geometrical notions, such as those Euclid takes as primitive, are not taken to have a solid mathematical foundation until they are reduced to discrete, arithmetical ones. Consequently, modern philosophical work on the concrete and the intuitive in mathematics has neglected continuous geometric notions, and focused on the discrete, arithmetical notions central to modern foundational accounts. I aim in my research to correct this with the formal account of **Eu** as my starting point.

Questions to be investigated

Foundational issues

By the standard conception, a foundation for mathematics and/or a mathematical subject cleanly separates mathematical content from mathematical reasoning. Mathematical content is distilled into axioms that list the fundamental truths of the subject. Mathematical reasoning is the deduction of theorems from the axioms by pure logic: inferences depend only on the logical form of either the axioms or what has already been deduced from the axioms. In contrast, in my analysis of Euclid's proof method, mathematical content is not separated from Euclid's mathematical reasoning. Correct application of its diagrammatic inference schemes requires an understanding of certain geometric concepts. Yet the inference schemes only play a role in simple settings, where the inferences at issue are evidently correct or incorrect. It is thus not obvious why instances of the scheme cannot form a part of a rigorous geometric proof.

These considerations suggest that geometric intuition has a constrained yet substantive role to play in the mathematical foundations of geometry. In my future research I plan to pursue this possibility along both technical and philosophical paths. In the technical direction, the goal is to deepen our understanding of the model theory of geometric intuition. The analysis of [Avigad et al, 2010] reveals that **Eu** could be interpreted in terms of models like those developed for Kant's transcendental logic in [Achourioti and van Lambalgen, 2011]. At the same time, Ken Manders in [Mancosu, 2008] draws a connection between Euclid's diagrammatic method (as analyzed by **Eu**) and models for the theory of real closed fields. I aim in future work to clarify precisely how **Eu** ought to be understood in relation to these two kinds of models.

On the philosophical side, the goal is to develop an epistemology of geometric intuition on the basis of the **Eu** formalism. The fundamental idea is to link the spatiality of **Eu** diagrams with the spatiality of geometric concepts. Accordingly, I will develop an account whereby geometric diagrams provide the means for the perception of spatial invariants (in some sense of perception) and thus provide a kind of perceptual warrant in the proofs of elementary geometry. On a parallel track, I will explore how the issue of elementary geometry's consistency is transformed when the theory is understood in terms of **Eu**. As **Eu** is a formal system, the issue does not disappear. But the concrete, spatial character of its diagrams promise new ways for examining, and perhaps resolving, the issue.

The **Eu** analysis of Euclid is relevant not only to informal proof in elementary geometry, but to informal proof in mathematics generally. It provides, specifically, an example of a *quasi-formal* proof method. The method is informal in that mathematical content has an irreducible justificatory role in inference. At the same time it is subject to formal constraints. Specifically, the representations that ground content-based inferences are formally specified, and the inferences so-grounded must satisfy conditions stated in terms of the formalism. (For more on the notion of a quasi-formal method, see 'The role of geometric content in Euclid's diagrammatic reasoning' available on my website www.johnmumma.org. A French version of the paper appears in *Les Etudes Philosophiques* 2011 no. 2.)

The general epistemological question I plan to investigate is the viability of an alternative foundational model where quasi-formal proofs are permitted. The ideal of proof guiding this investigation will be a Cartesian one, whereby all proof steps are consciously registered and the soundness of each step is evident. The standard foundational model seems sufficient with respect to this ideal. The issue I aim to explore is whether it is necessary—i.e. can the ideal be achieved with a foundation that allows mathematical content into proofs? Addressing the question requires a thorough study of the relationship between mathematical concepts and mathematical norms of inference. My analysis of Euclid's elementary geometry provides a sharp picture of a specific case. In attacking the question, I also plan to consult work in the formal verification of mathematical proofs. As pointed out in [Avigad et al., 2007], proof steps that are immediately evident in the informal setting of mathematical practice often expand into a staggering number of lines in a formal verification. The question in such cases is whether anything general can be said about what underlies the basic status of the informal inferences. In particular, do the mathematical concepts involved with the inferences play a role similar to that played by geometric concepts in the **Eu** analysis of Euclid?

Even if it turns out that a viable foundational model cannot be built from such investigations, the project would still have been worthwhile. It would in this case lead to a deeper understanding of the prominence of the standard foundational model.

Ontological issues

Developing an account of the relation between Euclid's mathematical concepts and his norms of inference requires of course an account of Euclid's mathematical concepts. What exactly is Euclid's elementary geometry about as a mathematical theory? Ever since modern axiomatizations of the subject appeared at the end of the 19th century, the dominant tendency has been to assimilate the subject into the rest of mathematics. Accordingly, there is nothing distinctive about Euclid's mathematical theory with respect to its ontology.

This is no by means immediate if we understand Euclid's theory according **Eu**. Specifically, though we can interpret the formalism of **Eu** in the same way we interpret modern axiomatic theories of geometry, it is not immediate that we have to do so. Alternative interpretations, more intuitive and geometric in character, seem possible. By them, Euclid's theory would not be about what modern axiomatizations are taken to be about—i.e. a certain kind of *space* or *structure*. Instead, it would be about *constructed geometric figures*. The former interpretation presumes a completed infinite totality, and arguably allows for abstract (i.e. non-spatial) objects. The objects of the latter interpretation are composed of a finite number of intrinsically spatial objects.

This interpretation is discussed in my paper 'Constructive Geometric Reasoning and Diagrams' (to appear in a special *Synthese* issue on diagrams in mathematics, and also available on my website). The specific purpose of the paper is show how **Eu** can be used to clarify what precisely is constructive about Euclid's geometry. That it is constructive is often asserted. Yet it is not immediately clear how the proofs of a fundamentally geometric (i.e. non-aritmetized) theory can be understood as such. In the paper I show how **Eu** and its diagrammatic inference schemes provide a promising way to do this.

I also in the paper highlight the difficulties in working such an interpretation out. The main challenge is accounting for the generality of Euclid's arguments without presuming an infinite space of points. Euclid's propositions have a general scope. A single proposition does not concern a single geometric configuration but a range of geometric configurations. The pressing question is the ontological status of the configurations in this range. The natural modern answer is to locate them in the abstract mathematical space characterized by the axioms for Euclidean geometry. I aim to explore the possibility of alternative answers. There is undoubtedly a significant epistemological difference between geometric figures in indefinite but bounded spaces and an unbounded geometric space. In exploring the viable alternatives for interpreting **Eu**, I will be exploring viability of understanding this difference in ontological terms.

I plan in doing so to draw upon the rich amount of material on the subject in the work of the major early modern philosophers. As these thinkers worked in the period before modern axiomatizations of geometry, there is much in their writings relevant to the relation of geometric figures to geometric space. A primary figure, of course, is Kant, who formulated influential doctrines on both the role of figures in geometric proofs and the concept of space. Leibniz is also relevant in this regard, as demonstrated resoundingly in [DeRisi 2007]. A specific historical project I am undertaking along these lines is a study of the works of Berkeley and Thomas Reid. In his *An Inquiry into the Human Mind on the Principles of Common Sense*, Reid sought to refute George Berkeley's claims in *A New Theory of Vision* by arguing that visual space had its own special non-Euclidean geometry, 'the geometry of visibles.' Though the explicit issue between the two philosophers is the nature of vision, they also seem to instantiate, if only implicitly, the contrast between the ancient and modern conceptions of geometry. Berkeley's philosophical doctrines clearly commit him to the classical conception of geometry as a theory of spatial figures. The notion of an underlying, imperceptible geometrical space is far too abstract for him to accept as legitimate. In contrast Reid, in responding to Berkeley's doctrines, seems to have employed the notion: for Reid there is a structure ('the geometry of visibles') that lies behind visual objects ('visibles') and accounts for their behavior.

Cognitive issues

In [Leitgeb, 2009], Hannes Leitgeb proposes a cognitive theory of mathematical proof based on the premise that 'the semantic and intuitive components of mathematically proofs are epistemically interdependent.' I plan to investigate how my analysis of the *Elements* can be thought to support and advance such a theory. **Eu** is not purely diagrammatic. Results are justified by both sentential and diagrammatic symbols. The distinction between these two kinds of symbols in the *Element*s would seem to link naturally to Leitgeb's general distinction between the semantic and intuitive components of a proof. Accordingly, what **Eu** isolates as sentential in Euclid's proofs would fall on the semantic side of Leitgeb's distinction, while what they isolate as diagrammatic would fall on the intuitive side.

The main questions involved in working this correspondence out are the following. What is nature of the intuitive (i.e. non-propositional) mental representations that ground the inferences that are instances of the inference scheme of my analysis? What is the relationship of these internal representations to the external symbols (both diagrammatic and sentential) that comprise an **Eu** derivation? How is the interaction between the non-propositional and propositional representations in an **Eu** derivation to be understood?

The project would thus draw upon and be relevant to psychological work on spatial cognition and symbolic representation. It would be relevant to the philosophy of mathematics as a case study. A sharp picture of the interdependence between the semantic and intuitive components of proofs in elementary geometry could perhaps serve as a model for understanding the interdependence in other areas of mathematics.

A question I will be sensitive to in pursuing this project is the efficacy of geometric diagrams in economizing geometric thought. The general point made by [Avigad et al., 2007] referred to above seems to find a vivid illustration in elementary geometry. Formal propositional proofs in the subject appear maddeningly prolix in comparison to their informal diagrammatic versions. The cognitive work required to take in the former seems far greater than that required to take in the latter. Furthermore, this extra work seems unnecessary. The extra details that one must attend to in the propositional proof do not seem essential to establishing the result. If Leitgeb's general picture can be worked out with **Eu**, an explanation for this would be readily available. The excessive detail of a purely propositional proof is the result of artificially jamming the intuitive component of a mathematical proof into propositional form. But even if turns out that geometric diagrams ought to be understood in propositional terms, there would still be questions to address with respect to how diagrams encode propositions, and their apparent efficacy in doing so.

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